RJBS
Romanian Journal of Building Services
Revista Română de Instalații
Volume 2 / No. 1 / 2015 / www.rjbs.ro

Heat and wave-like equations with variable coefficients solved by Taylor series method

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Abstract: *This paper presents a method for solving heat-like and wave-like equations with variable coefficients. The method is based on the Taylor expansion of analytical functions and transforms the PDE together with the initial/boundary conditions into a system of linear equations. Applications of the method are illustrated by three numerical examples.*

Key words : Taylor series, Initial/boundary-value problems, Solution of partial differential equations

1. Introduction

Most physical phenomena described by functions which depend on two or more independent variables can be modeled by partial differential equations [1]. Heat and wave-like equations are second-order partial differential equations of parabolic and, respectively, hyperbolic type which arise in scientific models like fluid mechanics, propagation of shallow water waves, vibrating string and chemical reaction-diffusion models. In this paper we present a numerical method for solving one-dimensional heat-like equations (1) and wave-like equations (2):

$$\frac{\partial u}{\partial t} - \alpha(x,t)\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad (1)$$

$$\frac{\partial^2 u}{\partial t^2} - \alpha(x,t)\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad (2)$$

where $\alpha(x,t)$ is a positive function. Taylor expansion method transforms the partial differential equation into a set of linear equations whose unknowns are the coefficients of the Taylor series expansion of the searched function [2]. The method can be applied for initial-value or boundary-value problems when the solution is analytic in order to obtain a polynomial approximation of the solution, or even the exact solution (when the limit of the polynomial sequence can be found - see the numerical applications).

2. Taylor series method

If $\Omega \subset \mathbf{R}^2$ is an open set which contains the origin $(0,0)$ and $u:\Omega \to \mathbf{R}$ is an analytic function on Ω, then it can be expanded in power series as follows (see [2-4]):

$$u(x,t) = \sum_{i,j=0}^{\infty} u_{i,j} x^i t^j \tag{3}$$

The partial derivatives of the function $u(x,t)$, $\frac{\partial^{m+n} u}{\partial x^m \partial t^n} \overset{not}{=} u^{(m,n)}(x,t)$ can be also expanded in power series as:

$$u^{(m,n)}(x,t) = \sum_{i,j=0}^{\infty} u_{i,j}^{(m,n)} x^i t^j \text{ , } \forall m,n \geq 0 \text{,} \tag{4}$$

where $u^{(0,0)}(x,t) = u(x,t)$ and $u_{i,j}^{(0,0)} = u_{i,j}$, $\forall i, j \geq 0$.

With the above notations, the following recurrence relations can be proved [4]:

$$u_{i,j}^{(m+1,n)} = (i+1) u_{i+1,j}^{(m,n)}, \tag{5}$$

$$u_{i,j}^{(m,n+1)} = (j+1) u_{i,j+1}^{(m,n)}, \tag{6}$$

for any $i, j, m, n \geq 0$.

Now, let us approximate $u(x,t)$ by the Taylor polynomial of degree *N*, $P_N(x,t)$. Consequently, we take $u_{i,j} = 0$ for all i, j s.t. $i + j > N$, so we can write:

$$u(x,t) \approx \overline{u}(x,t) = P_N(x,t) = X^T A T, \tag{7}$$

where $X = \left(1, x, x^2, \ldots, x^N\right)^T$, $T = \left(1, t, t^2, \ldots, t^N\right)^T$ and $A = \left(u_{i,j}\right)_{i,j=\overline{0,N}}$. Note that *A* is a triangular matrix.

The partial derivatives of *u* can be approximated by

$$u^{(m,n)}(x,t) \approx \overline{u}^{(m,n)}(x,t) = X^T A^{(m,n)} T, \tag{8}$$

where $A^{(m,n)} = \left(u_{i,j}^{(m,n)}\right)_{i,j=\overline{0,N}}$ and $A^{(0,0)} = A$.

By using the recurrence relations (5) and (6), we obtain that:

$$A^{(m,n)} = C^m A (C^n)^T, \tag{9}$$

where *C* is the matrix below:

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{10}$$

Now, let us write the matrices $A^{(0,1)}$, $A^{(2,0)}$ and $A^{(0,2)}$ used in the approximation of the partial derivatives that appear in our heat and wave-like equations:

$$u^{(0,1)}(x,t) \approx \bar{u}^{(0,1)}(x,t) = X^T A^{(0,1)} T \tag{11}$$

$$u^{(2,0)}(x,t) \approx \bar{u}^{(2,0)}(x,t) = X^T A^{(2,0)} T \tag{12}$$

$$u^{(0,2)}(x,t) \approx \bar{u}^{(0,2)}(x,t) = X^T A^{(0,2)} T \tag{13}$$

where

$$A^{(0,1)} = AC^T = \begin{pmatrix} u_{0,1} & 2u_{0,2} & 3u_{0,3} & \dots & (N-1)u_{0,N-1} & Nu_{0,N} & 0 \\ u_{1,1} & 2u_{1,2} & 3u_{1,3} & \dots & (N-1)u_{1,N-1} & 0 & 0 \\ u_{2,1} & 2u_{2,2} & 3u_{2,3} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ u_{N-2,1} & 2u_{N-2,2} & 0 & \dots & 0 & 0 & 0 \\ u_{N-1,1} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

$$A^{(2,0)} = C^2 A = \begin{pmatrix} 2u_{2,0} & 2u_{2,1} & 2u_{2,2} & \dots & 2u_{2,N-2} & 0 & 0 \\ 6u_{3,0} & 6u_{3,1} & 6u_{3,2} & \dots & 0 & 0 & 0 \\ 12u_{4,0} & 12u_{41} & 12u_{4,2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ N(N-1)u_{N,0} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

$$A^{(0,2)} = A(C^2)^T = \begin{pmatrix} 2u_{0,2} & 6u_{0,3} & 12u_{0,4} & \dots & n(n-1)u_{0,N} & 0 & 0 \\ 2u_{1,2} & 6u_{1,3} & 12u_{1,4} & \dots & 0 & 0 & 0 \\ 2u_{2,2} & 6u_{2,3} & 12u_{2,4} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 2u_{N-2,2} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \quad (16)$$

Now, let $\varphi(x,t) = X^T A T$ be a polynomial of degree at most $N-k$ with respect to *x* and at most $N-l$ with respect to *t*. Then

$$x^k t^l \varphi(x,t) = X^T D^k A \left(D^l\right)^T T, \quad (17)$$

where *D* is the matrix bellow:

$$D = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \quad (18)$$

Consider the following initial-value problem for heat-like equations with variable coefficients:

$$\frac{\partial u}{\partial t} - \alpha(x,t)\frac{\partial^2 u}{\partial x^2} = f(x,t) \quad (19)$$

$$u(x,0) = g(x), \quad (20)$$

Suppose that

$$f(x,t) = \sum_{i,j=0}^{\infty} f_{i.j} x^i t^j, \;\; \alpha(x,t) = \sum_{i,j=0}^{\infty} \alpha_{i.j} x^i t^j, \;\; g(x) = \sum_{i=0}^{\infty} g_i x^i \quad (21)$$

are analytic functions. Then, from Cauchy-Kovalevskaya Theorem, it follows that the above initial-value problem has a unique local analytic solution [2, 5]. Let us use the Taylor polynomials (around the origin) to approximate the known functions $f(x,t)$, $\alpha(x,t)$, $g(x)$ as well as the unknown function $u(x,t)$:

$$u(x,t) \approx P_N(x,t) = \sum_{i=1}^{N} \sum_{j=1}^{N-i} u_{i,j} x^i t^j \tag{22}$$

By replacing in the partial differential equation (19) and in the initial condition (20), we obtain a system of linear equations whose unknowns are the coefficients $u_{i,j}$.

3. Numerical aplications

Example 1. Consider the following heat-like equation

$$\frac{\partial u}{\partial t} - \frac{x^2}{6} \cdot \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbf{P}, \quad t > 0 \tag{23}$$

with the initial condition:

$$u(x,0) = x^3, x \in \mathbf{P} \tag{24}$$

We search for $u(x,t) \approx \overline{u}(x,t) = P_N(x,t) = X^T A T$, where $X = (1, x, \ldots, x^N)^T$, $T = (1, t, \ldots, t^N)^T$ and $A = (u_{i,j})_{i,j=\overline{0,N}}$ is the matrix of the coefficients ($u_{i,j} = 0$ for $i + j > N$). By applying the formulas (11), (12) and (17) we can write

$$\frac{\partial \overline{u}}{\partial t} = X^T A^{(0,1)} T, \quad \frac{\partial^2 \overline{u}}{\partial x^2} = X^T A^{(2,0)} T \text{ and } x^2 \frac{\partial^2 \overline{u}}{\partial x^2} = X^T D^2 A^{(2,0)} T,$$

so the equation (23) can be written in the matrix form:

$$A^{(0,1)} = \frac{1}{6} D^2 A^{(2,0)} \tag{25}$$

and we find $u_{0,j} = 0, u_{1,j} = 0,\ j \geq 1$ and the linear equations:

$$(j+1) u_{i,j+1} = \frac{i(i-1)}{6} u_{i,j}, \quad j \geq 0, \ i \geq 2 \tag{26}$$

Now, from the initial condition (24) we have:

$$\overline{u}(x,0) = \sum_{i=0}^{N} u_{i,0} x^i = x^3 \Rightarrow u_{i,0} = \begin{cases} 1, \text{if } i = 3 \\ 0, \text{if } i \neq 3 \end{cases} \tag{27}$$

From (26) and (27) we obtain that $u_{i,j} = \begin{cases} \frac{1}{j!}, \text{if } i = 3 \\ 0, \text{if } i \neq 3 \end{cases}$, so the n^th^ degree Taylor polynomial approximation of the unknown function $u(x,t)$ is

$$u(x,t) \approx x^3 + x^3 t + x^3 \frac{t^2}{2!} + x^3 \frac{t^3}{3!} + \ldots + x^3 \frac{t^{N-3}}{(N-3)!}.$$

When $N \to \infty$ we obtain $u(x,t) = x^3 e^t$, the exact solution of the problem.

Example 2. Consider the following initial-value problem for the wave-like equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{2} \cdot \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbf{P}, \quad t > 0 \tag{28}$$

$$u(x,0) = x + x^2, x \in \mathbf{P}, \tag{29}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, x \in \mathbf{P}. \tag{30}$$

We search for $u(x,t) \approx \overline{u}(x,t) = P_N(x,t) = X^T A T$, where $X = (1, x, \ldots, x^N)^T$, $T = (1, t, \ldots, t^N)^T$ and $A = (u_{i,j})_{i,j=\overline{0,N}}$ is the matrix of the coefficients ($u_{i,j} = 0$ for $i + j > N$). By applying the formulas (12), (13) and (17) we obtain that

$$\frac{\partial^2 \overline{u}}{\partial t^2} = X^T A^{(0,2)} T, \ \frac{\partial^2 \overline{u}}{\partial x^2} = X^T A^{(2,0)} T \text{ and } x^2 \frac{\partial^2 \overline{u}}{\partial x^2} = X^T D^2 A^{(2,0)} T,$$

so the equation (28) can be written in the matrix form:

$$A^{(0,2)} = \frac{1}{2} D^2 A^{(2,0)} \tag{31}$$

and we obtain that $u_{0,j} = 0, u_{1,j} = 0,$ for all $j \geq 2$ and the linear equations:

$$(j+1)(j+2) u_{i,j+2} = \frac{i(i-1)}{2} u_{i,j}, \ j \geq 0, \ i \geq 2. \tag{32}$$

Now, from the initial condition (29) we have:

$$\overline{u}(x,0) = \sum_{i=0}^{N} u_{i,0} x^i = x + x^2 \Rightarrow u_{i,0} = \begin{cases} 1, \text{if } i = 1 \text{ or } i = 2 \\ 0, \text{if } i \neq 1,2 \end{cases} \tag{33}$$

and the initial condition (30) can be written:

$$\frac{\partial \overline{u}}{\partial t}(x,0) = \sum_{i=0}^{N-1} u_{i,1} x^i = 0 \Rightarrow u_{i,1} = 0, i \geq 0 \tag{34}$$

The relations (32) and (34) implies $u_{i,2k+1}=0$ for $i\geq 2$ and $k\geq 0$. From (32) and (33) we find:

$$u_{i,2k}=\begin{cases}\frac{1}{(2k)!}, \text{if } i=3\\ 0, \text{if } i\neq 3\end{cases},$$

so the n[th] degree Taylor polynomial approximation of the solution $u(x,t)$ is

$$u(x,t)\approx x+x^2+x^2\frac{t^2}{2!}+x^2\frac{t^2}{4!}+\ldots+x^2\frac{t^{N-2}}{(N-2)!}.$$

When $N\to\infty$ we obtain $u(x,t)=x+x^2\cosh t$, the exact solution of the problem.

Example 3. Now, let us consider the following boundary-value problem:

$$t^2\frac{\partial^2 u}{\partial t^2}=x^2\frac{\partial^2 u}{\partial x^2},\quad x\in(0,1), t>0 \tag{35}$$

with the initial condition:

$$u(x,0)=1, x\in[0,1] \tag{36}$$

and the boundary conditions:

$$u(0,t)=1, t>0 \tag{37}$$

$$u(1,t)=e^t, t>0 \tag{38}$$

We search for the polynomial approximation $u(x,t)\approx\overline{u}(x,t)=P_N(x,t)=X^TAT$. Since

$\frac{\partial^2\overline{u}}{\partial t^2}=X^TA^{(0,2)}T$, $\frac{\partial^2\overline{u}}{\partial x^2}=X^TA^{(2,0)}T$, $t^2\frac{\partial^2\overline{u}}{\partial t^2}=X^TA^{(0,2)}\left(D^2\right)^TT$ and $x^2\frac{\partial^2\overline{u}}{\partial x^2}=X^TD^2A^{(2,0)}T$,

the equation (35) can be written in the matrix form:

$$A^{(0,2)}\left(D^2\right)^T=D^2A^{(2,0)} \tag{39}$$

and we obtain $u_{i,0}=0, u_{i,1}=0,$ for $i\geq 2$ $u_{0,j}=0, u_{1,j}=0,$ for $j\geq 2$ and the linear equations:

$$j(j-1)u_{i,j}=i(i-1)u_{i,j}\text{, for } i,j\geq 2\text{, so } u_{i,j}=0 \text{ for } i\neq j,\ i,j\geq 2. \tag{40}$$

From the initial condition (36) and from the boundary condition (37) we have:

$$\bar{u}(x,0)=\sum_{i=0}^{N}u_{i,0}x^{i}=1\Rightarrow u_{i,0}=0 \text{ for all } i\geq 1 \text{ and } u_{0,0}=1 \tag{41}$$

$$\bar{u}(0,t)=\sum_{j=0}^{N}u_{0,j}t^{j}=1\Rightarrow u_{0,j}=0 \text{ for all } j\geq 1. \tag{42}$$

Hence, the Taylor polynomial has the following form:

$$u(x,t)\approx P_N(x,t)=1+\sum_{i=1}^{[N/2]}u_{i,i}x^{i}t^{i} \tag{43}$$

Now, let us use the last boundary condition (37):

$$u(1,t)=e^{t}=1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\approx 1+\sum_{i=1}^{[N/2]}u_{i,i}t^{i}\Rightarrow u_{i,i}=\frac{1}{i!} \text{ for all } i\geq 0,$$

so $u(x,t)\approx\sum_{i=0}^{[N/2]}\frac{x^{i}t^{i}}{i!}$. When $N\to\infty$ we obtain $u(x,t)=e^{xt}$, the exact solution of the problem.

4. Conclusions

In this paper we used the Taylor series to solve initial and boundary-value problems for heat and wave-like equations with two independent variables. The solution of partial differential equations is reduced to the problem of solving a system of linear equations whose unknowns are the coefficients of the Taylor series. The validity of the method is illustrated by numerical applications. The method can be extended to PDEs with three or more independent variables.

5. References

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